RESEARCHERID
THOMSON REUTERS
[COTII 2019]
ISSN 2348 – 8034
Impact Factor- 5.070

GLOBAL JOURNAL OF ENGINEERING SCIENCE AND RESEARCHES

DECOMPOSTION OF THE TENSOR T_{ijkh} IN A FINSLER SPACE

P.K . Dwivedi[*1], S.C. Rastogi[2], A.K. Dwivedi[3] & S.Maithani[4]
[*1&4]AIMT, sLUCKNOW
[2]Flat No. 105, Shagun Vatika, Lucknow
[3]CIPET, LUCKNOW

ABSTRACT

The-tensor Tijkh, was simultaneously defined and studied in an n-dimensional Finsler space by Matsumoto [3] and Kawaguchi [1], in (1972). This is one of the most important tensor in the study of Finsler spaces of n-dimensions and it has been studied by several authors namely Matsumoto and Shimada [5], Rastogi [7] and [9] and others. In this paper an attempt has been made to decompose this tensor and study some of its properties. Besides this we have also defined and studied an n-dimensional T-reducible Finsler space

I. INTRODUCTION

Let F^n be an n-dimensional Finsler space with metric function L(x, y), metric tensor g_{ij} (x, y), and angular metric tensor h_{ij} and torsion tensor $C_{ijk.}$ The h-and v-covariant derivatives of a tensor field X^i_j are defined as follows Rund [10]:

$$X^i_{j|k} = \delta_k X^m_j F^i_{mk} - X^i_m F^m_{jk} \qquad (1.1)$$

And

$$X^i_{j|k} = \Delta_k X^i_j + X^m_j C^i_{mk} - X^i_m C^m_{jk}, \qquad (1.2)$$

Where $\delta_k = \mathfrak{z}_k = N^m_k \Delta_m,\ \mathfrak{z}_k$ and Δ_k respectively denote partial differentiation with respect to x^i and y^i.

The two torsion tensors A_{ijk} and P_{ijk} are defined as

$$A_{ijk} = L\, C_{ijk},\ 2\, C_{ijk} = \Delta_k\, g_{ij},\ P_{ijk} \quad A_{ijk|0} = A_{ijk|}{}^r\, l^r = y^i L^{-1} \qquad (1.3)$$

The second and third curvature tensors are given as

$$P_{ijkh} = C_{(i,j)} \{ A_{jkh|i} + A_{jkr} P^r{}_{jh} \} \qquad (1.4)$$

And

$$S_{ijkh} = C_{(k.h)} \{A_{ihr} A^r{}_{jh}\}$$

Where $C_{(i,j)}$ mean interchange of indices I and j and subtraction.

The T-tensor is symmetric in i, j, k, h and is expressed as Matsumoto [4]:

$$T_{ijkh} = L\, C_{ijk}\|_h + 1_i C_{jkh} + 1_j C_{khi} + l_k C_{ijh} + 1_h C_{ijk} \qquad (1.6)$$

II. DECOMPOSITION OF T-TENSOR IN $F^{n.}$ Let T_{ijkh} be expressed as

$$T_{ijkh} = M_{ijk} B_h + M_{ikh} B_j + M_{jkh} B_i + M_{ijh} B_k + C_{ij} C_{kh} + C_{ij} C_{ih} + C_{ik} C_{jh} + D_i D_j D_k D_h \qquad (2.1)$$

Where the vectors B_j and D_j are non-zero and the tensors M_{ijk} and C_{ij} are symmetric and non-zero.

If we define a tensor L_{ijkh} by

$$L_{ijkh} = C_{ij} C_{kh} + C_{jk} C_{ih} + C_{ik} C_{jh} + D_i D_k D_h \qquad (2.2)$$

From equation (2.1) on multiplication by $1^{h,}$ we can obtain

$$M_{ijk} \; B_o + M_{iko} B_j + M_{ijo} B_k + L_{ijko} = 0. \qquad (2.3)$$

Equation (2.3) on further multiplication by 1^k gives

$$2 M_{ijo} B_o + M_{ioo} \; B_j + M_{jooo} B_i + L_{ijoo} = 0. \qquad (2.4)$$

Equation (2.4) on multiplication by I^j gives

$$3 M_{ioo} B_o + M_{ooo} B_i + L_{iooo} =0 \qquad (2.5)$$

Equation (2.5) on multiplication by 1^i gives

$$4 M_{ioo} B_o +3 C^2{}_{00} +D^4{}_0 \; = 0 \qquad (2.6)$$

If we assume that $B_o \neq 0$, equation (2.6) implies

$$M_{ooo} = \; - (4B_o)^{-2} (3C^2{}_{00} + D_0{}^4) \qquad (2.7)$$

Substituting the value of M_{000} from (2.7) in (2.5), we get

$$M_{i00} = (B_0)^{-2} [(3 C_{00}{}^2 + D_0{}^4) Bi – 4 B_0 L_{i000}] / 12 \qquad (2.8)$$

Substituting the value of M_{i00} from (2.8) in (2.4) we get

$$M_{ij0} = - (B_0)^{-3} [B_i B_j (3 C_{00}{}^2 + D_0{}^4) – 2 B_0 (B_j L_{i000} + B_i \; L_{j000})]/12 – (2 \; B_0)^{-1} L_{ij000} \qquad (2.9)$$

Substituting from equation (2.9) the value of M_{ij0} in (2.3), we get

$$M_{ijk} = (1/4)B_0{}^{-4} (3C_{00}{}^2 +D_0{}^4)B_i \; B_j B_k –(1/3) B_0{}^{-3} (B_i B_j L_{k000} + B_j b_k \; L_{i000} +B_k B_i L_{jooo})+(1/2) B_0{}^{-2} (B_i \; L_{jk00} + B_j \; L_{ij00} -2 B_0 \; L_{ijko}). \qquad (2.10)$$

Substituting in (2.1) from (2.2) and (2.10), we get on simplification

$$T_{ijkh} \; = L_{ijkh} \; - B_0{}^{-1} \; (B_i L_{jkho} \; + B_j \; L_{ikh0} + B_h L_{ijk0}) + B_0{}^{-2} (B_i B_j L_{kh00} + B_j \; B_k \; L_{hi00} \; + B_k B_h L_{ij00} \; + B_h B_i L_{jk00} + B_j \; B_k L_{jh00} + B_j \; B_h \; L_{ik00}) – B_0{}^{-3} (B_i \; B_j \; B_k L_{h000} + B_j B_k B_h L_{i000} + B_k B_h B_i \; L_{j000} + B_h B_i B_j L_{k000}) + B_0{}^{-4} (3 C_{00}{}^2 + D_0{}^4) B_i B_j B_k B_h. \qquad (2.11)$$

From equation (2.11), we can establish:

Theorem 2.1- In an n-dimensional Finsler space F^n , if $B_0 \neq 0$, the tensor T_{ijkh} can be decomposed in the form of (2.11).

If in equation (2.6) we assume that $M_{000} = 0$ and $B_0 \neq 0$, we get $C_{00}{}^2 + D_0{}^4 = 0$, which leads to:

Theorem 2.2- If $B_0 \neq 0$, the necessary and sufficient condition for vanishing for M_{000} is given by $3 C_{00}^2 + D_0^4 = 0$. Substituting $M_{000} = 0$, in (2.3), (2.4) and (2.5) we get

$$M_{i00} = D_0^3 (3 B_0 C_{00})^{-1} (C_{i0} D_0 - C_{00} D_i) \quad (2.12)$$

$$M_{ij0} = -D_0^3 (6B_0^2 C_{00})^{-1}[B_i (C_{j0} D_0 - C_{00} D_j) + B_j (C_{i0} D_0 - C_{00} D_i)] - (2B_0)^{-1} (C_{ij} C_{00} + C_{i0} C_{j0} + D_i D_j D_0^2) \quad (2.13)$$

$$M_{ijk} = -(B_0)^{-1}\Sigma_{(i,j,k)}[-D_0^3(6B_0^2 C_{00})^{-1} \{B_i(C_{j0} D_0 - C_{00} D_j) + B_j (C_{i0} D_0 - C_{00} D_i)\} - (2B_0)^{-1} (C_{ij} C_{00} + C_{i0} C_{j0} + D_i D_j D_0^2) + C_{ij} C_{k0} + (1/3)D_i D_j D_k D_0] \quad (2.14)$$

Application of (2.12), (2.13) and (2.14) in (2.11) gives.

$$T_{ijkh} = L_{ijkh} - B_0^{-1} (B_i L_{jkh0} + B_j L_{ikh0} + B_k L_{ijh0} + B_h L_{ijk0}) + B_0^{-2}(B_i B_j L_{kh00} + B_j B_k L_{hi00} + B_k B_h L_{ij00} + B_h B_i L_{jk00} + B_i B_k L_{jh00} + B_j B_h L_{ik00}) - B_0^{-3} (B_i B_j B_k L_{h000} + B_j B_k L_{h000} + B_j B_k B_h L_{i000} + B_k B_h B_i L_{j000} + B_h B_i B_j L_{k000}), \quad (2.15)$$

Where

$$L_{000} = 0,\ L_{i000} = 3C_{00} D_0^{-1} (C_{i0} D_0 - C_{00} D_i),$$
$$L_{ij00} = C_{00} D_0^{-2} (C_{ij} D_0^2 - 3 C_{00} D_i D_j) + 2 C_{i0} C_{j0},$$
$$L_{ijk0} = C_{ij} C_{k0} + C_{jk} C_{i0} + C_{ik} C_{j0} - 3 D_0^{-3} C_{00}^2 D_i D_j D_k \quad (2.16)$$

Hence we have:

Theorem 2.3- In an n-dimensional Finsler space F^n, if $B_0 \neq 0$ and $M_{000} = 0$, the tensor T_{ijkh} can be expressed by (2.15).

III. T- REDUCIBLE FINSLER SPACES

We shall now consider some special cases

Case I. $C_{ij} = h_{ij}$: Equation (2.1), by virtue of (2.2) can be expressed as

$$T_{ijkh} = h_{ij} h_{kh} + h_{jk} h_{ih} + h_{ik} h_{jh} + D_i D_j D_k D_h \quad (3.1)$$

Furthermore from equation (3.1), by virtue of $T_{ijkh} 1^h = 0$, we can obtain, $D_0 = 0$. It is very well known that torsion vector C_i satisfies $C_0 = 0$, therefore most suitable value of the tensor T_{ijkh} can be expressed as

$$T_{ijkh} = h_{ij} h_{kh} + h_{jk} h_{ih} + h_{ik} h_{jh} + C_i C_j C_k C_h \quad (3.2)$$

From equation (3.2), we give following definition:

Definition 3.1- A finsler space F^n, whose tensor T_{ijkh} is given by (3.2), shall be called T-reducible Finsler space.

Case II. Two-dimensional Finsler space F^2: Equation (3.2), can be expressed as

$$T_{ijkh} = (3+C^4) m_i m_j m_k m_h \quad (3.3)$$

It is known that in a two dimensional Finsler space T_{ijkh} is expressed as Matsumoto [4]

$$T_{ijkh} = L^{-1} I_{,2} m_i m_j m_k m_h,$$

ISSN 2348 – 8034
Impact Factor- 5.070

Therefore, comparing equations (3.3) and (3.4), we obtain

Theorem 3.1- In a two dimensional T-reducible Finsler space scalar $l,_2 = L\,(C^4+3)$.

Case III. Three dimensional Finsler space F^3: Equation (1.6) can be expressed as Rastogi [9]

$T_{ijkh} = m_i m_j m_k \acute{\alpha}_h - \Sigma_{(i,j,k)}\{m_i m_j m_k n_h \beta_h = m_i m_j n_k \gamma_h\} + n_i n_j n_k \delta_h$ (3.5)

Where

$\acute{\alpha}_h = L\, C_{(1)}\|_h + C_{(1)} l_h + 3\, C_{(2)} v_h$, $\beta_h = L\, C_{(2)}\|_h + C_{(2)} l_h - (C_{(1)} - 2\, C_{(3)}) v_h$.
$\gamma_h = L\, C_{(3)}\|_h + C_{(3)} l_h + C_{(2)} v_h$. $\delta_h = L\, C_{(2)}\|_h + C_{(2)} l_h$ (3.6)

In a three dimensional finsler space equation (3.2) can be expressed as

$T_{ijkh} = (3+C^4)\, m_i m_j m_k m_h + 3\, n_i n_j n_k n_h + \Sigma_{(i,j,k)}\{m_i m_j n_k n_h + n_i n_j m_k m_h\}$ (3.7)

Comparing equations (3.5) and (3.6), we can obtain

$L\, C_{(1)}\|_0 = -C_{(1)}$, $L\, C_{(1)}\|_h m^h = 3\,(1 - C_{(2)}\, v_{2)32}) + C^4$, $L\, C_{(1)}\|_h n^h = -3\, C_{(2)}\, v_{2)33}$,
$L\, C_{(2)}\|_0 = -C_{(2)}$, $L\, C_{(2)}\|_h m^h = (C_{(1)} - 2C_{(3)} v_{2)32}$, $L\, C_{(2)}\|_h n^h = (C_{(1)} - 2C_{(3)})\, v_{2)33} - l$
$L\, C_{(3)}\|_0 = -C_{(3)}$, $L\, C_{(3)}\|_h m^h = -C_{(2)} v_{2)32} l$, $L\, C_{(3)}\|_h n^h = -C_{(2)} v_{2)33}$ (3.8)

Hence we have:

Theorem 3.2- In a three dimensional T-reducible Finsler space F^3, coefficients $C_{(1)}\|_0$ and $C_{(3)}\|_0$ satisfy equations (3.8).

IV. T-REDUCIBLE N-DIMENSIONAL FINSLER SPACES

Here we shall consider some special cases of n-dimensional finsler spaces

Case I. C2 – like Finsler space: It is known that a C2 – like Finsler space satisfies [6]

$C_{ijk} = C^{-2}\, C_i C_j C_k$ (4.1)

Therefore, from equation (1.6), we can obtain

$T_{ijkh} = L\{-2\, C^{-3} C\|_h\, C_i C_j C_k + C^{-2}(C_i\|_h\, C_k + C_i C_j C_k\|_h)\}$
$+ C^{-2}\,(l_i C_j C_k C_h + l_j C_k C_i C_h + l_k C_i C_j C_h + l_h C_i C_j C_k)$ (4.2)

If C2 –like Finsler space is also T-reducible Finsler space, comparing equations (3.2) and (4.2) and multiplying the resulting equation by g^{ij}, we get

$(n+1)\, h_{kh} + C^2 C_k C_h = L\{-2C^{-1} C\|_h C_k + 2\, C^{-2} C_i\|_h C^i C_k + C_k\|_h\} + l_k C_h + l_h C_k$ (4.3)

Which further leads to

$C^i (C_i\|_h\, C_k - C_i\|_k\, C_h) = C(C\|_h C_k - C\|_h C_k - C\|_k C_h)$, $C^i C_i\|_0 = -L^{-1} C^2$ (4.4)
Hence we have:

Theorem 4.1- An n-dimensional C2-like T-reducible Finsler space F^n, satisfies (4.4)

Case II. P2-like Finsler space: It is known that for an arbitarary vector field M_i, the second curvature tensor of a P2 – like Finsler space satisfies [4]

$$P_{ijkh} = M_i C_{jkh} - M_j C_{ikh} \qquad (4.5)$$

Which leads to $P_{jkh} = M_0 C_{jkh}$. Using this relationship in equation (1.6), we obtain on simplification

$$L\, T_{ijkh}\|_0 - M_0 T_{ijkh} = L(L\, C_{ijk}\|_h\|_o - C_{ijk}\|_h) \qquad (4.6)$$

If P2 –like Finsler space F^n is also T-reducible, by virtue of equations (3.2) and (4.6), we can obtain on simplification

$$L(L\, C^h\|_{h\,|0} - C^h\|_h) = C^3(4L\, C\|_0 - C\, M_0) - (n^2-1) M_0 \qquad (4.7)$$

Hence we have:

Theorem 4.2- An n-dimensional P2-like T-reducible Finsler space F^n, satisfies (4.7).

Case III. PT2-like Finsler space: It is known that in a PT2-like Finsler space tensor P_{ijk} satisfies Rastogi [8]:

$$P_{ijk} = P^{-2} P_i P_j P_k \qquad (4.8)$$

By virtue of equations (1.6), (3.2) and (4.8), we can easily obtain

$$L^2 C^h\|_{h|0} = 2 C^2 (C\, P\|_0 + P^h C_h) \qquad (4.9)$$

Hence we have:

Theorem 4.3- An n-dimensional PT2-like T-reducible Finsler space F^n, satisfies (4.9)

Case IV. C-Reducible Finsler space: It is known that in a C-reducible Finsler space T_{ijkh} is given by Matsumoto [2]:

$$T_{ijkh} = L\,(n^2 - 1)^{-1} C^r\|_r \Sigma_{(I,\, j,\, k)} \{h_{ij} h_{kh}\} \qquad (4.10)$$

Therefore on comparison with equation (3.2) , we can obtain

Theorem 4.4- If an n-dimensional C-reducible Finsler space F^n, is also T-reducible, it satisfies $L\, C^r\|_r = C^4 + n^2 - 1$.

Case V. P-reducible Finsler space: It is known that in a P-reducible Finsler space F^n

$$P_{ijk} = (n+1)^{-1} (A_i\|_0 h_{jk} + A_{j|0} h_{ki} + A_k\|_0 h_{ij}) \qquad (4.11)$$

Which by virtue of equation (3.2) and

$$T_{ijkh}\|_0 = L^2 C_{ijk}\|_{h|0} + l_0 P_{jkh} + l_j P_{kih} + l_k P_{ijh} + l_h P_{ijh} \qquad (4.12)$$

On simplification leads to

$$C^h\|_{h|0} = 4 C^2 L^{-2} A_{h|0} C^h \qquad (4.13)$$

Hence we have:

Theorem 4.5- If an n-dimensional P-reducible Finsler space F^n is also T-reducible, it satisfies equation (4.13).

Case VI. T2-like Finsler space: It is known that in a T2- like Finsler space F^n $(n > 2)$, T_{ijkh} is expressed as Rastogi [7]

$$T_{ijkh} = A(x,y)\, \Sigma_{(I,\, j,\, k)}\{h_{ij}\, h_{kh}\} \qquad (4.14)$$

Therefore from equations (3.2) and (4.14), we can obtain

$$A(x,y) = (C^4 + n^2 - 1)(n^2-1)^{-1} \qquad (4.15)$$

Which leads to?

Theorem 4.6- In a T2-like Finsler space F^n, which is also T-reducible, the scalar $A(x,y)$ is given by equation (4.15).

Case VII. T3-like Finsler space: It is known that in a T3- like Finsler space F^n $(n > 3)$, T_{ijkh} is expressed as Rastogi [9]

$$T_{ijkh} = \Sigma_{(I,\, j,\, k)}\{\, a_{hk}\, h_{ij} + b_{hk}\, C_i\, C_j\} \qquad (4.16)$$

Where a_{hk} and b_{hk} are arbitrary second order tensors such that $a_{ho} = 0$ and $b_{ho}=0$.

Comparing equations (3.2) and (4.16), we can obtain on simplification

$$(n+1)\, a_{ok} + (C^2\, \delta^i_k + 2\, C^i\, C_k)\, b_{0i} = 0 \qquad (4.17)$$

Hence we have:

Theorem 4.7- In a T3-like Finsler space F^n, which is also T-reducible, the tensors a_{0k} and b_{ok} satisfy equation (4.17).

Case VIII. $A_{hk} = P_{hk}^{(1)}$: It is known that $P_{hk}^{(1)}$, is given by Shimada [11]

$$P_{hk}^{(1)} = C_{k|h} - C^j_{hk|j} + P^j_{kr}\, C^r_{jh} - P^r_{hk}\, C_r \qquad (4.18)$$

Therefore from equations (3.2) and (4.18), we can obtain

$$B_{hk}\, C^k = (3C^2)^{-1}[(n+1+C^4)C_h - (n+1)\{C_{k|h} - C^j_{hk|j} + P^j_{kr}\, C^r_{jh} - P^r_{\ hk}\, C_r\}\, C^k] \quad (4.19)$$

Which by virtue of

$$(n+1)(P_{hk}^{(1)} - h_{hk}) + b_{hk}\, C^2 + 2\, b_{hi}\, C^i\, C_k = C^2\, C_k\, C_h, \qquad (4.20)$$

Leads to

$$B_{hk} = C^{-2}[(n+1)(h_{hk} - P_{hk}^{(1)}) + (3C^2)^{-1}\, C_k\, C_h(3\, C^4 - 2C^2 - 2(n+1) + 2C_k\, \{(n+1)(3C^2)^{-1} + (C_{i|h} + C^j_{hi|j} - P^j_{ir}\, C^r_{jh} + P^r_{hi}\, C_r)C^i\}] \qquad (4.21)$$

Hence we have:

Theorem 4.8- In a T3-like Finsler space F^n, which is also T-reducible and from which the tensor a_{hk} is equal to $P^{(1)}_{hk}$, the tensor b_{hk} is given by (4.21).

REFERENCES

1. *Kawaguci, H, On Finsler spaces with the vanishing second curvature tensor, Tensor, N.S., 26(1972), 250-254.*
2. *Matusmote, M.: On C-reducible Finsler space, Tensor, N.S., 24(1972), 29-37.*
3. *Matsumoto, M.: V-transformations of Finsler spaces, I, Definitions, infinitesimal transformations and isometrics', J.Math. Kyoto Univ., 12(1972), 479-512.*
4. *Matsumoto, M.: Foundations of Finsler Geometry and special finsler space, Kaiseish, Press, Otsu, Japan, 1986.*
5. *Matsumoto, M and Shimada, H.: On Finsler spaces with the curvature tensor* P_{hijk} *and* S_{hijk} *satisfying special conditions, Rep. on Math. Physics, 12(1977), 77-87.*
6. *Rastogi, S.C.: and Kawasguchi, H.: A geometrical meaning of the P-reducible confection in Finsler spaces, Tensor, N.S., 51(1992), 251-256.*
7. *Rastogi, S.C.: T2 and TR2-like Finsler spaces, J. Nat, Acad, Math., 17(2003)m 1-8.*
8. *Rastogi, S.C.: On certain P-reducible Finsler spaces, Ganita56, I(2005), 55-64.*
9. *Rastogi, S.C.: T3-like Finsler spaces General society 2(2008) 49-65.*
10. *Rund, H.: The differential geometry of Finsler spaces, Springer – Verlag, 1959.*
11. *Shimada, H.: On the Ricci tensors of Particular Finsler spaces, J. Korean math. Soc., 14, 1 (1977), 41-63.*